68

Computational problems of the hierarchical aggregation of OWA operators

Vincenzo CUTELLO
Department of Mathematics
University of Catania
Catania, Italy

Javier MONTERO
Faculty of Mathematics
Complutense University
Madrid, Spain

***Abstract*— In this paper we will analyze some problems related with ordered hierarchical aggregations of OWA operators as defined in [4]. In particular, we will provide polynomial algorithms to maximize or minimize the dispersion and in some cases the orness of the hierarchical OWA aggregation.**

Key words: Aggregation rules, fuzzy preferences, decision making.

I. Introduction and Preliminaries

In some recent scientific investigations the notion of hierarchical aggregation of information has been introduced along with some characterization theorems (see [3,4]). Intuitively, hierarchical aggregations are aggregations of chunks of information which in turn represent aggregated information. The practical consequences of such hierarchical aggregations are quite interesting. If we have aggregation maps of very big dimensions, hierarchical aggregations will allow us to deal with sub-aggregation operators of smaller dimensions, whose computational jobs can be parallelized. Thus, it will be possible to obtain a significant speed up of the whole aggregation process.

By considering the natural ordering on the input real values we can talk about "ordered" hierarchical aggregation rules. This type of hierarchical aggregations allows in a very natural way to deal with Yager's OWA operators.

II. OWA Operators

Ordered Weighted Averaging (OWA) operators were proposed by Yager in [7]. Since their introduction they have been used and applied to many fields, such as Neural Networks, Database Systems, Learning Systems and Fuzzy Logic Controllers (see [8] for a comprehensive review on the subject). T-norms and T-conorms represent aggregation operators that generalize the notion of conjunction and disjunction of classical logic. The min operator is the maximal T-norm and the max operator is the minimal T-conorm (see [6].) OWA operators fill the gap between min and max. So, intuitively by means of OWA operators we can go from conjunction (intersection) to disjunction (union) in a continuous way.

To simplify the formalization of OWA operators let us introduce the notion of *sorting permutation* of a list.

If $L = [a_1, a_2, \dots, a_n]$ is a list of numbers, a sorting permutation σ for L is any permutation of the elements of L that produces a list

$$\sigma(L) = [a_{[1]}, \dots, a_{[n]}]$$

verifying

$$a_{[i]} \geq a_{[j]}$$

for all $i \leq j$.

So, if for example $L = [0.2, 0.1, 0.3]$, then we shall have $\sigma(L) = [0.3, 0.2, 0.1]$.

DEFINITION 1 *An OWA operator of dimension n is an aggregation operator ϕ that has an associated list of weights $W = [w_1, \dots, w_n]$ such that*

1. $w_i \in [0,1]$ *for all* $1 \leq i \leq n$

2. $\sum_{i=1}^{n} w_i = 1$

3. *for any* $L = [a_1, a_1, \ldots, a_n]$ *and its corresponding* $\sigma(L) = [a_{[1]}, \ldots, a_{[n]}]$, *it is defined*

$$\phi(L) = \sum_{i=1}^{n} w_i a_{[i]}.$$

□

In view of the above definition, it can be immediately verified that OWA operators are commutative, monotone and idempotent.

Two significative measures are associated with OWA operators of dimension n.

(m1) The first measure is called *orness* and it estimates how close an OWA operator is to the max operator. It is defined as

$$orness(\phi) = \frac{1}{n-1} \sum_{i=1}^{n} (n-i) w_i.$$

Dual to the measure of *orness* is the measure of *andness* defined as

$$andness(\phi) = 1 - orness(\phi)$$

which therefore measures how close an OWA operator is to the min operator.

(m2) The second measure called *dispersion* estimates the degree to which all aggregates are used equally, i.e. the degree to which an OWA operator is close to the simple average operator. The dispersion measure is defined as

$$Disp(\phi) = -\sum_{i=1}^{n} w_i \ln w_i.$$

III. Ordered Hierarchies of OWA Operators

We will now give a characterization of hierarchical aggregations of OWA operators that produce OWA operators.

Let $\phi_0, \phi_1, \ldots, \phi_c$ be $c + 1$ OWA operators such that

- ϕ_0 has dimension c;
- ϕ_i has dimension h_i for any $i = 1, 2, \ldots, c$
- $\sum_{i=1}^{c} h_i = n$

Let $w_{0,1}, \ldots, w_{0,c}$ be the weights associated to ϕ_0, and for all $i = 1, \ldots, c$ let $w_{i,1}, \ldots, w_{i,h_i}$ be the weights associated to ϕ_i.

DEFINITION 2 *The* ordered hierachical composition *of* $\phi_0, \phi_1, \ldots, \phi_c$ *is defined by*

$$\begin{aligned} &\phi_0(\phi_1, \ldots, \phi_i, \ldots, \phi_c)(a_1, \ldots, a_n) = \\ &= \phi_0(\ldots, \phi_i(a_{[H_{i-1}+1]}, \ldots, a_{[H_i]}), \ldots) \end{aligned}$$

for all n-uples $(a_1, \ldots, a_n)$, *where* $H_i = \sum_{j=1}^{i} h_j$. □

DEFINITION 3 *If a property* P *that holds for* $\phi_0, \phi_1, \phi_2, \ldots, \phi_c$ *holds for the ordered hierarchical aggregation* $\phi_0(\phi_1, \ldots, \phi_c)$ *as well, we will then say that* P propagates under ordered hierarchical aggregations. □

As proven in [4] we have

THEOREM 1 *The property of being an OWA operator propagates under ordered hierarchical aggregations.* ■

Moreover, in view of theorem 1 the following properties propagate under ordered hierarchical aggregations of OWA operators:

- Monotonicity
- Idempotency
- Commutativity

IV. Orness and Dispersion

Given a fixed ordered hierarchical aggregation

$$\phi \equiv \phi_0(\phi_1, \ldots, \phi_c)$$

based upon $c + 1$ OWA operators, Yagers's orness measure takes the expression

$$\begin{aligned} orness(\phi) &= \phi(\frac{n-1}{n-1}, \frac{n-2}{n-1}, \ldots, \frac{1}{n-1}, 0) = \\ &= \phi_0(z_1, \ldots, z_i, \ldots, z_c) \end{aligned}$$

where each z_i takes the value

$$\phi_i\left(\frac{n-(\sum_{j=1}^{i-1}h_j+1)}{n-1},\ldots,\frac{n-\sum_{j=1}^{i}h_j}{n-1}\right)=$$
$$=\sum_{m=1}^{h_i}\frac{n-(\sum_{j=1}^{i-1}h_j+m)}{n-1}w_{i,m}=$$
$$=\frac{n-\sum_{j=1}^{i}h_j}{n-1}+\frac{\sum_{m=1}^{h_i}(h_i-m)w_{i,m}}{n-1}=$$
$$=\frac{n-\sum_{j=1}^{i}h_j}{n-1}+(h_i-1)\frac{orness(\phi_i)}{n-1}$$

for all $i=1,\ldots,c$, with h_i the dimension of each ϕ_i and $(w_{i,1},\ldots,w_{i,h_i})$ its associated weights. Hence,

$$orness(\phi)=$$
$$=\sum_{i=1}^{c}\frac{(h_i-1)orness(\phi_i)+n-\sum_{j=1}^{i}h_j}{n-1}w_{0,i}$$

Obviously, changes in ϕ_j's weights for $j=1,\ldots,c$ in such a way that $orness(\phi_j)$ does not decrease for any $j=1,\ldots,c$ will never make $orness(\phi)$ decrease. However, increasing $orness(\phi_0)$ does not necessarely increase $orness(\phi)$ as shown by the following example.

Example 1 Let ϕ_0,ϕ_0' be two OWA operators of dimension c such that their associated weights are, respectively, $w_{0,2}=1, w_{0,i}=0$ for all $i\neq 2$ and $w_{0,1}'=w_{0,3}'=1/2, w_{0,i}'=0$ for all $i\neq 1,3$. Then, $orness(\phi_0)=orness(\phi_0')$. Let now ϕ_i be the minimum rule (i.e., $orness(\phi_i)=0$ for all $i=1,\ldots,c$). Then the difference between $orness(\phi_0(\phi_1,\ldots,\phi_c))$ and $orness(\phi_0'(\phi_1,\ldots,\phi_c))$ will still depend on the relative sizes h_2 and h_3 of OWA operators ϕ_2,ϕ_3. □

The dispersion of the hierarchical aggregation can be computed as follows.

Denote by ϕ an OWA operator obtained from an ordered hierarchical aggregation of OWA operators $\phi_0,\phi_1,\ldots,\phi_c$ with dimensions $c,h_1,\ldots,h_c$ respectively. Clearly,

$$Disp(\phi)=-\sum_{i=1}^{h_1}w_{0,1}w_{1,i}\ln[w_{0,1}w_{1,i}]-\ldots$$
$$-\sum_{i=\alpha_{c-1}+1}^{n}w_{0,c}w_{c,i-\alpha_{c-1}}\ln[w_{0,c}w_{c,i-\alpha_{c-1}}]=$$
$$=-\sum_{j=1}^{c}\sum_{i=1}^{h_j}(w_{0,j}w_{j,i-\alpha_{j-1}}\ln w_{0,j}+$$
$$+w_{0,j}w_{j,i-\alpha_{j-1}}\ln w_{j,i-\alpha_{j-1}})=$$
$$=-\sum_{j=1}^{c}w_{0,j}\ln w_{0,j}-\sum_{j=1}^{c}w_{0,j}Disp(\phi_j)=$$
$$=Disp(\phi_0)+\sum_{j=1}^{c}w_{0,j}Disp(\phi_j)$$

In both cases it is clear that fixed the ϕ_0 and the other c OWA operators, the ornes and dispersion values of the hierarchical aggregation depend upon the particular ordering of the c OWA operators. It is then natural to ask the following questions:

(1) How can one find an ordering of those c OWA aggregated operators which maximizes [resp. minimizes] the dispersion value of the hierarchical aggregation?

(2) How can one find an ordering of those c OWA aggregated which maximizes [resp. minimizes] the orness value of the hierarchical aggregation?

In the next section we will answer the above questions.

V. Maximizing and Minimizing Dispersion and Orness

Let ϕ_0 be an OWA operator of dimension c and let $\Phi=\{\phi\}$ be a set of OWA operators of cardinality c. Moreover, let π be a permutation of the indices $\{1,2,\ldots,c\}$ such that

$$i\leq j \text{ if and only if } w_{0,\pi(i)}\geq w_{0,\pi(j)}.$$

Intuitively, for $i=1,2,\ldots,c$, $w_{0,\pi(i)}$ is the i-th weight in non-increasing order.

Let $\phi_1,\ldots,\phi_c$ be an ordering of the OWA operators in Φ and let $h_1,\ldots,h_c$ their respective dimensions. We will now prove that the following lemmas hold.

LEMMA 1 *The value*

$$Disp(\phi)=Disp(\phi_0)+\sum_{j=1}^{c}w_{0,j}Disp(\phi_j)$$

is maximum if and only if the ordering $\phi_1, \ldots, \phi_c$ *verifies*

$$i \leq j \text{ if and only if } Disp(\phi_{\pi(i)}) \geq Disp(\phi_{\pi(j)}).$$

■

LEMMA 2 *The value*

$$Disp(\phi) = Disp(\phi_0) + \sum_{j=1}^{c} w_{0,j} Disp(\phi_j)$$

is minimum if and only if the ordering $\phi_1, \ldots, \phi_c$ *verifies*

$$i \leq j \text{ if and only if } Disp(\phi_{\pi(i)}) \leq Disp(\phi_{\pi(j)})$$

■

In order to prove the two lemmas above, we make use of the elementary result that if $a \geq b$ and $c \geq d$ then $ac + bd \geq ad + bc$

(L1) Let us then prove Lemma 1.

Suppose first that $Disp(\phi)$ is maximum and suppose by contradiction that there exist i, j with $i < j$ such that

$$Disp(\phi_{\pi(i)}) < Disp(\phi_{\pi(j)})$$

Then, since

$$w_{0,\pi(i)} Disp(\phi_{\pi(j)}) + w_{0,\pi(j)} Disp(\phi_{\pi(i)}) >$$
$$> w_{0,\pi(i)} Disp(\phi_{\pi(i)}) + w_{0,\pi(j)} Disp(\phi_{\pi(j)})$$

by switching ϕ_i and ϕ_j we would obtain an ordering which would cause the dispersion of the hierarchical aggregation to increase, contradicting the initial hypothesys.

Conversely, suppose that $\phi_1, \ldots, \phi_c$ verifies the condition of the Lemma. Let $\phi'_1, \ldots, \phi'_c$ be an ordering which maximizes $Disp(\phi)$. From what above proven, this ordering verifies the condition of the Lemma as well.

Let us prove that

$$\sum_{j=1}^{c} w_{0,j} Disp(\phi_j) = \sum_{j=1}^{c} w_{0,j} Disp(\phi'_j).$$

If $\phi_{\pi(1)} \neq \phi'_{\pi(1)}$ in view of the condition of the Lemma it must be $Disp(\phi_{\pi(1)}) = Disp(\phi'_{\pi(1)})$. Thus, we are left to prove that

$$\sum_{j \neq \pi(1)} w_{0,j} Disp(\phi_j) = \sum_{j \neq \pi(1)} w_{0,j} Disp(\phi'_j).$$

Thus, by applying a recursive argument we prove our claim.

(L2) Lemma 2 can be proven analogously.

In view of the above Lemmas and from the well-known result (see, e.g., [1]) that any given n numbers can be sorted in $\mathcal{O}(n \log n)$ time, we can claim that the following theorem is true.

THEOREM 2 *Given an OWA operator* ϕ_0 *of dimension* c *and a set of* c *OWA operators* Φ *and given their dispersions, there exists a* $\mathcal{O}(c \log c)$ *algorithm which produces an ordering of* Φ *maximizing (or minimizing) the dispersion of the hierarchical aggregation.* □

The problem of maximizing or minimizing the orness of the hierarchical aggregation appears to be quite more difficult from a computational point of view. Indeed, it is not difficult to find similarities with some classical decision problems which are computationally hard (see [5]). Therefore, even though the problem may turn out not to be *NP*-hard, we expect a polynomial solution (if any) to be certainly not trivial.

$\mathcal{O}(c \log c)$ algorithms can instead be given for the most significative cases in practice. Since each one of those c OWA operators are just sumarizing partial information to be aggregated into only one index, it is common to define them in such a way that either

- they contain the same amount of information. In this case, all OWA operators in Φ will have the same dimension $h = n/c$; or
- they treat their inputs with the same degree of optimism (pessimism). In this case, all OWA operators will have the same degree of orness.

Let us consider the above two cases in more detail.

(case 1) In case all OWA operators in Φ have the same dimension $h = n/c$, we have to find an ordering $\phi_1, \ldots, \phi_c$ such that

$$orness(\phi) = \sum_{i=1}^{c} \frac{(h-1)orness(\phi_i) + n - \sum_{j=1}^{i} h}{n-1} w_{0,i}$$

is maximum. Therefore,

$$orness(\phi) = = \frac{h-1}{n-1} \sum_{i=1}^{c} orness(\phi_i) w_{0,i} + + \frac{h}{n-1} \sum_{i=1}^{c} (c-i) w_{0,i}.$$

It follows that the sought ordering can be obtained as in Lemma 1, i.e. by producing any ordering which verifies $i \leq j$ if and only if

$$orness(\phi_{\pi(i)}) \leq orness(\phi_{\pi(j)}).$$

Analogously, if we want to minimize the orness of the hierarchical aggregation we act as in Lemma 2.

(case 2) In case all OWA operators in Φ have the same orness r, we have

$$orness(\phi) = \frac{1}{n-1} \sum_{i=1}^{c} (r h_i + \sum_{j=i+1}^{c} h_j) w_{0,i} - \frac{r}{n-1} = = \frac{1}{n-1} \sum_{i=1}^{c} (r w_{0,i} + \sum_{j=1}^{i-1} w_{0,j}) h_i - \frac{r}{n-1}.$$

As a consequence, any ordering which sorts the elements of Φ in increasing [resp. decreasing] order with respect to their dimensions maximizes [resp. minimizes] the orness.

VI. Final Comments

In this paper, we continue our research work aimed at a full charaterization of hierarchical aggregation operators (see [2,3]) by focussing our attention on the hierarchical aggregations of OWA operators (see [4]). Since the property of being an OWA operator propagates under ordered hierarchical aggregations and since the orness and dispersion of the obtained OWA operator depends upon the given ordering of the aggregated OWA operators, we have studied how these values can be maximized or minimized. In particular, we have provided $O(c \log c)$ algorithms which solve the general problem for the dispersion and the most important cases in practice for the orness.

Acknowledgment: This research has been partially supported by Dirección General de Investigación Científica y Técnica (Spanish national grant PB91-0389).

References

[1] T.H. Cormen, C.E. Leiserson and R.R. Rivest *Introduction to Algorithms.* MIT Press, Cambridge, MA, 1990.

[2] V. Cutello and J. Montero. A characterization of rational amalgamation operations. *International Journal of Approximate Reasoning*, 8:325-344, 1993.

[3] V. Cutello and J. Montero. Hierarchies of intensity preference aggregations. *International Journal of Approximate Reasoning.* To appear.

[4] V. Cutello and J. Montero. Hierarchies of aggregation operators. *International Journal of Intelligent Systems* . To appear.

[5] M.R. Garey and D.S. Johnson. *Computers and Intractability. A guide to the theory of NP-Completeness.* W.H. Freeman and Co., New York , NY, 1979.

[6] G.J. Klir and T.A. Folger. *Fuzzy sets, Uncertainty and Information.* Prentice Hall, Englewood Cliffs, NJ, 1988.

[7] R.R. Yager. On ordered weighted averaging aggregation operators in multi-criteria decision making. *IEEE transactions on systems, man and cybernetics*, 18:183–190, 1988.

[8] R.R. Yager. Families of OWA operators. *Fuzzy Sets and Systems.* 59:125–148, 1993.

ACTES / PROCEEDINGS

volume I

CINQUIEME CONFERENCE INTERNATIONALE
FIFTH INTERNATIONAL CONFERENCE

IPMU

TRAITEMENT D'INFORMATION ET GESTION D'INCERTITUDES DANS LES SYSTEMES A BASE DE CONNAISSANCES

*INFORMATION **P**ROCESSING AND **M**ANAGEMENT OF **U**NCERTAINTY IN KNOWLEDGE-BASED SYSTEMS*

Paris, 4 - 8 Juillet 1994
Paris, July 4 - 8, 1994
Cité Internationale Universitaire

ISBN 2-905433-26-4